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Stability of the laminar mixing of two parallel streams with respect to supersonic disturbances

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The stability with respect to 'supersonic' disturbances of the laminar mixing of two parallel streams of a compressible fluid is studied. For locally supersonic disturbances there will be waves propagating outward from the mixing layer. The numerical results show that the flow is generally unstable with respect to supersonic disturbances, although the amplification rate is smaller than that for subsonic disturbances. The flow is more unstable at lower Mach number, and the increase of the angle between the disturbance wave-number vector and the principal flow direction tends to increase the instability.

1. Introduction

In a previous paper the stability with respect to subsonic disturbances of the laminar mixing of two parallel streams of a compressible fluid was studied (Lessen, Fox & Zien 1965*a*). For infinite Reynolds numbers, neutrally stable eigenvalues along with further indications of instability were obtained, and it was concluded that increasing the Mach number of the flow has a de-stabilizing effect. Furthermore, the flow is more unstable as the angle between the disturbance wave-number vector and the principal flow direction becomes larger. Since the time of the investigation by Lees & Lin (1946) of the stability of compressible flows, supersonic disturbances to a continuous flow profile have hardly been studied. For a discontinuous flow profile it was shown recently that supersonic disturbances may cause instability (Miles 1958; Lessen, Fox & Zien 1965*b*), but the amplification rate of the disturbance is relatively smaller than that of a subsonic disturbance.

For the case of supersonic disturbances, one can see from the asymptotic solutions to the linearized disturbance equation that progressive waves will be propagated obliquely to the viscous (continuously varying) layer of the streams. In the case of laminar mixing of two streams (half jet), one would expect only outgoing waves for stability considerations (Lees & Lin 1946; Miles 1958). The amplitudes of the disturbances are oscillatory but finite at infinite distance. Therefore, in a similar manner as for subsonic disturbances, one is able to obtain the relevant eigenvalues for the stability of the flow with respect to supersonic disturbances.

2. Basic equations

Consider two parallel semi-finitely extended streams of a viscous compressible fluid. The upper stream moves initially with a velocity u^* and the lower stream is initially at rest. Assume that the flow is subjected to a small disturbance having the form

$$Q = q(y)\exp[i(\alpha x + \beta z - \alpha c t)]. \tag{1}$$

For infinite Reynolds number, the differential equation for the Squire transformed-pressure disturbance $\tilde{\pi}$ is given in dimensionless form by (Lessen *et al.* 1965*a*)

$$\frac{d^2\tilde{\pi}}{dY^2} - \frac{2}{\bar{u}-c}\frac{d\bar{u}}{dY}\frac{d\tilde{\pi}}{dY} - \tilde{\alpha}^2\bar{T}[\bar{T} - \tilde{M}^2(\bar{u}-c)^2]\tilde{\pi} = 0, \tag{2}$$

where Y is the Howarth transformed co-ordinate

$$Y = \int_0^y (1/\bar{T})\,dy, \tag{3}$$

$\bar{u}$ the mean flow velocity with u^* as a reference quantity, $\bar{T}$ the mean temperature, $c\ (= c_r + ic_i)$ the complex wave velocity, $\tilde{\alpha}(=\alpha/\cos\Theta)$ the wave-number in the direction of wave propagation $\tilde{M}(=\tilde{M}_0\cos\Theta)$ the transformed Mach number, M_0 the Mach number of the upper stream, and Θ the angle between the wave-number vector and the mean flow direction.

The asymptotic solutions for $\tilde{\pi}$ are

$$\tilde{\pi} = A\exp\{-\tilde{\alpha}\bar{T}(-\infty)[1-\tilde{M}^2c^2/\bar{T}(-\infty)]^{\frac{1}{2}}Y\} + B\exp\{\tilde{\alpha}\bar{T}(-\infty)[1-\tilde{M}^2c^2/\bar{T}(-\infty)]^{\frac{1}{2}}Y\} \quad \text{as} \quad Y \to -\infty. \tag{4}$$

$$\tilde{\pi} = C\exp\{\tilde{\alpha}[1-\tilde{M}^2(1-c)^2]^{\frac{1}{2}}Y\} + D\exp\{-\tilde{\alpha}[1-\tilde{M}^2(1-c)^2]^{\frac{1}{2}}Y\} \quad \text{as} \quad Y \to +\infty. \tag{5}$$

For supersonic disturbances, i.e. disturbances whose wave velocity relative to the local flow, in the direction of wave propagation, has a magnitude greater than the speed of sound, one has

$$\tilde{M}^2(\bar{u}-c)^2/\bar{T} > 1. \tag{6}$$

Thus, the disturbances are oscillatory at large distance from the mixing layer, and behave as progressive waves. In distinguishing the outgoing and the incoming waves, one has to note that an incoming wave relative to a fixed coordinate may appear to be an outgoing wave to an observer moving with the free-stream velocity (Lees & Lin 1946). To a fixed observer, the first term in (4) or (5) represents an outgoing wave and the second term the incoming wave. It has been mentioned that we only expect an outgoing wave relative to the local fluid motion in the mixing of two streams. The second term in (4) drops out for locally supersonic disturbances with the lower stream at rest.

3. Neutral disturbances

When the disturbances are supersonic relative to the lower stream but subsonic relative to the upper stream, there will be a phase shift for the disturbances between $Y = -\infty$ and the point where $\bar{u} = c$; there will be no phase shift from the point where $\bar{u} = c$ to $Y = +\infty$. It is seen from the consideration of subsonic disturbances (table 1, Lesson *et al.* 1965*a*) that the foregoing is the only possible case for the existence of neutral disturbances at high Mach number and small wave-propagation angle relative to the mean flow. In this case the asymptotic behaviours of $\tilde{\pi}$ are

$$\left.\begin{aligned} \tilde{\pi} &= A\exp\{-i\tilde{\alpha}\bar{T}(-\infty)[\tilde{M}^2c^2/\bar{T}(-\infty)-1]^{\frac{1}{2}}Y\} \quad \text{as} \quad Y\to-\infty,\\ \tilde{\pi} &= D\exp\{-\tilde{\alpha}[1-\tilde{M}^2(1-c)^2]^{\frac{1}{2}}Y\} \quad \text{as} \quad Y\to+\infty,\end{aligned}\right\} \tag{5'}$$

where the coefficient A can be complex.

Let

$$G = \dot{\tilde{\pi}}/(\tilde{\alpha}^2\bar{T}\tilde{\pi}), \tag{7}$$

where the superscript 'dot' denotes the derivative of a quantity with respect to Y. Then equation (2) becomes

$$\dot{G} = [\bar{T}-\tilde{M}^2(\bar{u}-c)^2]+[2\dot{\bar{u}}/(\bar{u}-c)-\dot{\bar{T}}/\bar{T}]\,G-\tilde{\alpha}^2\bar{T}G, \tag{8}$$

and the boundary conditions for G are

$$\left.\begin{aligned} G &= -i[\tilde{M}^2c^2/\bar{T}(-\infty)-1]^{\frac{1}{2}}/\tilde{\alpha} \quad \text{at} \quad Y=-\infty,\\ G &= -[1-\tilde{M}^2(1-c)^2]^{\frac{1}{2}}/\tilde{\alpha} \quad \text{at} \quad Y=+\infty.\end{aligned}\right\} \tag{9}$$

Since there is a singularity at $\bar{u} = c$, we shall integrate equation (8) along the complex path as we did for the case of subsonic disturbances. However, no necessary and sufficient condition for the existence of neutral supersonic disturbances has been obtained. Thus, in order to satisfy the boundary condition, one has to try different sets of values of $\tilde{\alpha}$ and c.

4. Amplified or damped disturbances

When $c_i \neq 0$, equation (4) becomes

$$\tilde{\pi} \sim \exp[\pm i\tilde{\alpha}\bar{T}(-\infty)\{|[a+(a^2+b^2)^{\frac{1}{2}}]^{\frac{1}{2}}|+ib[a+(a^2+b^2)^{\frac{1}{2}}]^{\frac{1}{2}}\}\,Y/\sqrt{2}] \quad \text{for} \quad Y\to-\infty, \tag{10}$$

where
$$a = \tilde{M}^2(c_r^2-c_i^2)/\bar{T}(-\infty)-1,$$
and
$$b = 2\tilde{M}^2c_rc_i/\bar{T}(-\infty).$$

For positive c_i,

$$\tilde{\pi} \sim \exp[-i\tilde{\alpha}\bar{T}(-\infty)|\{a+(a^2+b^2)^{\frac{1}{2}}\}^{\frac{1}{2}}|\,Y/\sqrt{2}]\exp[\tilde{\alpha}\bar{T}(-\infty)|b\{a+(a^2+b^2)^{\frac{1}{2}}\}^{\frac{1}{2}}|\,Y/\sqrt{2}] \quad \text{for} \quad Y\to-\infty. \tag{11}$$

For negative c_i, an outgoing wave cannot exist, because otherwise $\tilde{\pi}$ becomes infinite at $Y = -\infty$. Therefore, the laminar mixing is usually unstable with respect to supersonic disturbances, and the neutral stability is an extreme case.

5. Numerical results and discussions

For the details of the method for numerical calculation, the readers are referred to the previous paper (Lessen *et al.* 1965*a*, or Zien 1965). Table 1 lists the eigenvalues for neutral supersonic disturbances relative to the lower stream. For comparison, eigenvalues for neutral subsonic disturbances are given outside the outlined box. In the numerical integration, the coefficient A in equation (5′)

was set to unity; thus, when the integration reached the real axis, the pressure disturbance became complex. By multiplying the disturbance function with the complex conjugate of the complex disturbance function at $Y = 6+0i$, we have a real disturbance function on the real axis from the singularity to $Y = +\infty$ (the constant A is made suitably complex). Figure 1 gives the transformed pressure disturbance along the complex path $Y_i = 0{\cdot}25\,Y_r - 1{\cdot}5$.

		Θ		
M_0		0°	30°	60°
1·5	$\tilde{\alpha}$	0·275310	0·294116	0·327533
	c	0·705676	0·705676	0·705676
2·0	$\tilde{\alpha}$	0·262274	0·276141	0·319798
	c	0·773659	0·766613	0·766613
3·0	$\tilde{\alpha}$	0·290855	0·303862	0·339668
	c	0·862356	0·859645	0·857686
4·0	$\tilde{\alpha}$	0·329587	0·342116	0·371337
	c	0·909946	0·909021	0·908493
5·0	$\tilde{\alpha}$	0·365500	0·376665	0·399755
	c	0·937360	0·937095	0·936996

TABLE 1. Wave-numbers and wave velocities for neutral disturbances at different Mach numbers and wave-propagation angles

		Θ		
M_0		0°	30°	60°
1·5	$(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$	$0{\cdot}5745-0{\cdot}6161i$	$0{\cdot}5417-0{\cdot}6232i$	$0{\cdot}4966-0{\cdot}6448i$
	$(\partial c/\partial\Theta)_{\tilde{\alpha},M_0}$	$0{\cdot}0000+0{\cdot}0000i$	$-0{\cdot}0312+0{\cdot}0343i$	$-0{\cdot}0255+0{\cdot}0329i$
	$(\partial c/\partial M_0)_{\Theta,\tilde{\alpha}}$	$0{\cdot}0677-0{\cdot}1200i$	$0{\cdot}0370-0{\cdot}1016i$	$0{\cdot}0027-0{\cdot}0672i$
2·0	$(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$	$0{\cdot}4806-0{\cdot}4085i$	$0{\cdot}5028-0{\cdot}4316i$	$0{\cdot}4616-0{\cdot}4724i$
	$(\partial c/\partial\Theta)_{\tilde{\alpha},M_0}$	$0{\cdot}0000+0{\cdot}0000i$	$-0{\cdot}0746+0{\cdot}0523i$	$-0{\cdot}0431+0{\cdot}0302i$
	$(\partial c/\partial M_0)_{\Theta,\tilde{\alpha}}$	$0{\cdot}0713-0{\cdot}0470i$	$0{\cdot}0583-0{\cdot}0732i$	$0{\cdot}0066-0{\cdot}0487i$
3·0	$(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$	$0{\cdot}2967-0{\cdot}2185i$	$0{\cdot}2842-0{\cdot}2286i$	$0{\cdot}2874-0{\cdot}2530i$
	$(\partial c/\partial\Theta)_{\tilde{\alpha},M_0}$	$0{\cdot}0000+0{\cdot}0000i$	$-0{\cdot}0204+0{\cdot}0100i$	$-0{\cdot}0164+0{\cdot}0145i$
	$(\partial c/\partial M_0)_{\Theta,\tilde{\alpha}}$	$0{\cdot}0115-0{\cdot}0264i$	$0{\cdot}0206-0{\cdot}0191i$	$0{\cdot}0049-0{\cdot}0179i$
4·0	$(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$	$0{\cdot}2031-0{\cdot}1501i$	$0{\cdot}2005-0{\cdot}1541i$	$0{\cdot}1732-0{\cdot}1533i$
	$(\partial c/\partial\Theta)_{\tilde{\alpha},M_0}$	$0{\cdot}0000+0{\cdot}0000i$	$-0{\cdot}0034+0{\cdot}0044i$	$-0{\cdot}0078+0{\cdot}0069i$
	$(\partial c/\partial M_0)_{\Theta,\tilde{\alpha}}$	$0{\cdot}0091-0{\cdot}0032i$	$0{\cdot}0041-0{\cdot}0064i$	$0{\cdot}0019-0{\cdot}0066i$
5·0	$(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$	$0{\cdot}1097-0{\cdot}1141i$	$0{\cdot}1124-0{\cdot}1118i$	$0{\cdot}1121-0{\cdot}1018i$
	$(\partial c/\partial\Theta)_{\tilde{\alpha},M_0}$	$0{\cdot}0000+0{\cdot}0000i$	$-0{\cdot}0049+0{\cdot}0030i$	$-0{\cdot}0039+0{\cdot}0035i$
	$(\partial c/\partial M_0)_{\Theta,\tilde{\alpha}}$	$0{\cdot}0032-0{\cdot}0032i$	$0{\cdot}0029-0{\cdot}0034i$	$0{\cdot}0008-0{\cdot}0023i$

TABLE 2. Partial derivatives of c at different Mach numbers and wave-propagation angles.

The rates of change of c with respect to $\tilde{\alpha}$, Θ and M_0 are given in table 2. The same conclusions as for subsonic disturbances can be drawn for supersonic disturbances. Namely, the flow is more unstable at lower Mach numbers, and the increase of the wave-propagation angle relative to the mean flow tends to destabilize the flow. However, in the present case, the wavelength of an ampli-

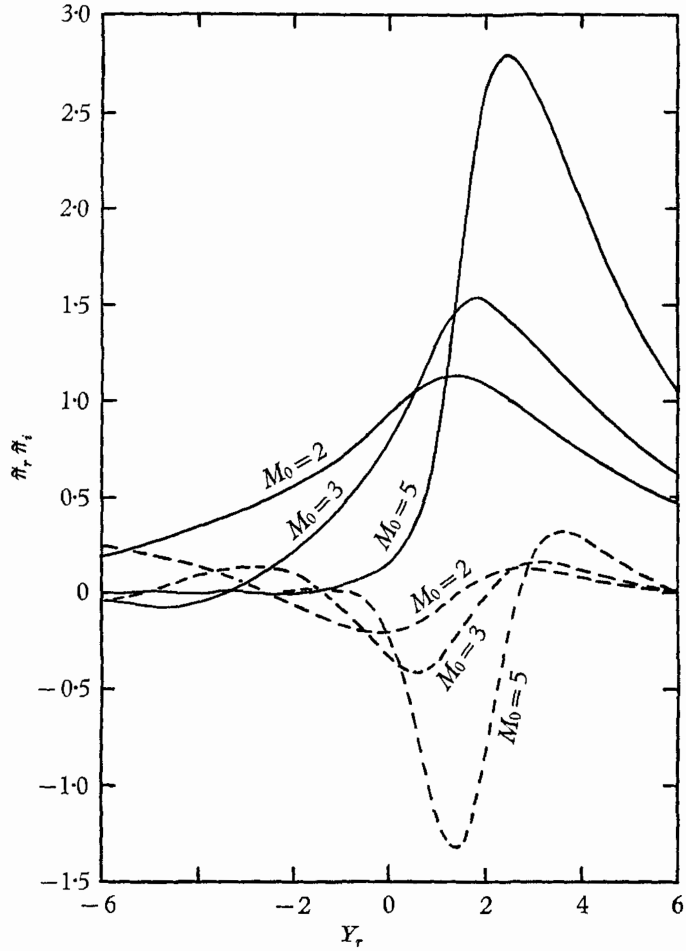

FIGURE 1. Pressure disturbance π (π_r, solid lines, and π_i, broken lines) along the complex path $Y_i = Y_r/4 - 1{\cdot}5$ at $\Phi = 0°$.

fied disturbance must be longer than that of a neutral disturbance, and no disturbance can exist with a wavelength shorter than the neutral wavelength for an outgoing wave at given Mach number and wave-propagation angle. Though the amplification rate for supersonic disturbances is relatively smaller than that for supersonic ones, the shear layer is generally unstable even at very high Mach number.

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